A Stochastic Lyapunov Theorem with Application to Stability Analysis of Networked Control Systems

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Abstract

The source of randomness in stochastic systems is an input with stochastic behavior as treated in the existing literature. Special types of stochastic processes such as the Wiener process or the Brownian motion have served as an adequate model of such an input for years. The body of stochastic systems theory is elegantly shaped around such input models. An example is the Itô's formula. With development of new applications, we are faced with various phenomena that are more demanding from a stochastic modeling approach.
To cope with this problem we restate the stochastic Lyapunov theorem such that it can be applied to a wider class of stochastic systems. In this paper stochastic systems are considered without imposing assumptions on the nature of the stochastic input and the way it affects the sample trajectories. Lyapunov stability theorem is represented for this type of systems in terms of a stability notion that generalizes the notion of stability in moments. As a result, the new theorem finds a larger domain of applications while it can be reduced to some known versions of the stochastic Lyapunov theorem. As an application, an existing deterministic result for nonlinear networked control systems is extended to a more practical probabilistic setting which extends the available analysis tools for checking the stability of continuous-time nonlinear networked control systems in the stochastic setting. The results are applied to a two-channel magnetic levitation system which is controlled over a local communication network to obtain a bound on the rate of transmission failures due to the presence of noise in the industrial environment.

Keywords: Networked Control Systems, Nonlinear Systems, Stochastic Systems, Lyapunov Theorem, Delayed Systems.

1. Introduction

In many applications, stability of a control system should be studied in presence of some random behavior. The theory of stochastic differential equations (SDEs) is used for this purpose [1,2,3,4]. An SDE can be regarded as a differential equation which depends on a stochastic process. The theory of SDEs is mainly developed for Itô SDE [4] and its applications to stochastic control problem are usually based on extensions of Lyapunov theorem [5,6,7]. There are problems that cannot be modeled using the Itô SDE like the random switched system in [8]. Another problem is the networked control system (NCS) analysis problem considered in this work. The difficulty is to model the stochastic phenomena as a Brownian motion to act as an input of the Itô SDE which is not always possible.

Several approaches have been used for handling NCS problems [9,10]. Two important issues are handling the stochastic effects such as communication delay and loss in the NCS [11]. LQG problem for linear NCSs with stochastic delays and packet losses is studies for example in [12,13]. An in-depth investigation of an NCS problem may lead to more detailed modeling and analysis, such as the relationship between stability and noise characteristics in [14], or network scheduling and topology related issues in [15,16]. Some basic works regarding nonlinear NCS are [17,18,19] where the network induced error is defined and modeled as a perturbation.

In this work, the Lyapunov stability method is presented in an abstract setting with respect to the Itô SDE stability analysis. In the Itô SDE, the usual source of randomness is an stochastic input. This input has certain properties that result in the Itô formula which is the basis of the related Lyapunov based stochastic stability analysis methods. However, our results are not based on the Itô formula which enables us to apply our results to an NCS problem (and possibly other new applications). For this purpose, a stability notion which is more suitable is used. Additional efforts may be required for applying the results to a problem. But, in return the results may be used for a wider class of applications. The main motivation of this work is its application in extending the NCS analysis performed in [17] in which the effect of shared communication on a nonlinear NCS is studied in a deterministic setting. The results of this paper are applied to obtain a practical probabilistic NCS analysis. Due to possibility of significant delays in an NCS, the Lyapunov results are presented for delayed systems to facilitate extension of the NCS analysis to delayed case in future. This paper is an enhanced version of [20] where the

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formulation of NCS analysis is improved and a new section is added to present an application of the results to a practical NCS problem.

In section two, the considered SDE is described. Also, the stability notions will be presented and their relations will be clarified. Section three contains the Lyapunov stability results. Extension of the NCS problem in [17] is studied in section four followed by an example. A practical case-study is studied in section five and conclusions are made at the end.

2. Preliminaries

Notation: Throughout the paper, the set of real numbers $(-\infty,\infty)$ is indicated by $\mathbb{R}$ and the set of non-negative real numbers $[0,\infty)$ is indicated by $\bar{\mathbb{R}}^+$ where ∞ is the positive infinity. Euclidian norm of a vector x is denoted by $\|x\|$. For an arbitrary set A, the set of mappings from A to $\mathbb{R}^n$ is denoted by $C^n(\mathsf{A})$ and the set of stochastic processes with sample paths in $C^n(\mathsf{A})$ is denoted by $\mathcal{B}^n(\mathsf{A})$. For $\varphi\in C^n(\mathsf{A})$, ∞-norm is defined as $\|\varphi\|_\infty = \sup_{\alpha\in\mathsf{A}}\|\varphi(\alpha)\|$. For $\varphi\in C^n(\mathbb{R})$ and $t\in\mathbb{R}$, history of φ at t denoted by $\varphi_t\in C^n(\bar{\mathbb{R}}^+)$ is defined as $\varphi_t(\alpha) = \varphi(t-\alpha)$ for any $\alpha\in\bar{\mathbb{R}}^+$. This convention is used to indicate an argument of a functional [21,22]. Accordingly, if $\psi\in\mathcal{B}^n(\mathbb{R})$ and $t\in\mathbb{R}$ then the history $\psi_t\in\mathcal{B}^n(\bar{\mathbb{R}}^+)$ can be defined as $\psi_t(\alpha) = \psi(t-\alpha)$. Probability of an event A is denoted by $\mathsf{P}(A)$.

2.1 The class of systems to be considered

The mathematical description of the class of systems considered in this paper is given by the stochastic functional differential equation (1) where $t\in\mathbb{R}$ is a time instant, $x(t)\in\mathbb{R}^n$ is the state vector, $\theta\in\mathcal{B}^m(\mathbb{R})$ and f: $C^n(\bar{\mathbb{R}}^+)\times\mathbb{R}\times\mathbb{R}^m\to\mathbb{R}^n$ is a functional.

$$\dot{x}(t) = f\big(x_t, t, \theta(t)\big) \tag{1}$$

Because of the randomness caused by θ, the state x is also a stochastic process. The initial time is denoted by t_0 up to which the state information is available.

The functional f is assumed to satisfy (2) for every $\varphi\in C^n$, $t\in\mathbb{R}$, $\theta\in\mathbb{R}^m$ which indicates that x=0 (or the origin) is an equilibrium solution of (1).

$$\|\varphi\|_\infty = 0 \quad\Rightarrow\quad f(\varphi,t,\theta) = 0 \tag{2}$$

Remark 2.1: The theory of stochastic differential equations (SDEs) is mainly concerned with the Itô SDE [4]. In an Itô SDE, $\theta(t)$ is basically a white noise process and f is affine with respect to θ. There are existence and uniqueness results for solution of Itô SDEs with delays [21].

Assumption 1: In this paper it will be assumed that (1) has a unique solution $x\in\mathcal{B}^n(\mathbb{R})$ for every initial conditions.

We are interested in determining the stability of (1) where the concept of stability is presented in the next part.

2.2 Definition of stability

Three important stability notions in the literature are stability in probability, stability in p-th moment and almost sure stability. In this paper we will work with a generalized version of stability in p-th moment, which is also related to stability in probability (definition 2.2 in the following).

Definition 2.1: A continuous function u: $\bar{\mathbb{R}}^+\to\bar{\mathbb{R}}^+$, $u(0)=0$, is said to belong to class K_d if it is non-decreasing and $u(\alpha)>0$ for $\alpha>0$. For u_1, $u_2\in K_d$ it is said that u_1 covers u_2 if there exist $c>0$ such that $u_1(\alpha)\geq c\,u_2(\alpha)$ for every $\alpha\in\bar{\mathbb{R}}^+$.

Definition 2.2: For a class K_d function h, the equilibrium x=0 of system (1) is *h-mean stable* if for any $\varepsilon>0$ there exist $\delta(\varepsilon,t_0)>0$ such that for any $t\geq t_0$ (3-1) holds. Moreover, x=0 is *asymptotically h-mean stable* if it is h-mean stable and there exists $\delta(t_0)>0$ such that (3-2) holds.

$$\|x_{t_0}\| < \delta \Rightarrow E\{h(\|x(t)\|)\} < \varepsilon \tag{3-1}$$

$$\|x_{t_0}\| < \delta \Rightarrow \lim_{t\to\infty} E\{h(\|x(t)\|)\} = 0 \tag{3-2}$$

Remark 2.2: If we select h as $h(\alpha)=\alpha^p$ for some $p>0$, definition of stability in p-th moment in [3] is retrieved. Stability in second moment or mean square stability, is a very practical stability concept specially for linear systems.

Definition of stability in probability from [3] with a few modifications to express it for (1) is as below.

Definition 2.3: The equilibrium x=0 of system (1) is *stable in probability* if for every pair ε_1, $\varepsilon_2>0$, there exists $\delta(\varepsilon_1,\varepsilon_2,t_0)>0$ such that (4-1) holds for any $t\geq t_0$. Also, x=0 is *asymptotically stable in probability* if it is stable in probability and for every $\varepsilon>0$, there exists $\delta(\varepsilon,t_0)>0$ such that (4-2) holds.

$$\|x_{t_0}\| < \delta \quad\Rightarrow\quad P\{\|x(t)\| \geq \varepsilon_1\} \leq \varepsilon_2 \tag{4-1}$$

$$\|x_{t_0}\| < \tilde{\delta} \quad\Rightarrow\quad lim_{t\to\infty} P\{\|x(t)\| \geq \tilde{\varepsilon}\} = 0 \tag{4-2}$$

Relationship between h-mean stability and stability in probability is stated as proposition 2.1 (proof is omitted).

Proposition 2.1: The system (1) is stable in probability if and only if there exists a K_d function h such that (1) is h-mean stable. Moreover, (1) is asymptotically stable in probability if there exists a K_d function h such that (1) is asymptotically h-mean stable.

Remark 2.3: Any of the above stability properties is said to be global when the value of related functions δ or δ can be made arbitrarily large by adjusting their first argument. Moreover, a stability property is said to be uniform if the related functions δ or δ are independent from t_0 (similar to the delay-free deterministic case in [23]).

Remark 2.4: The stability property of a stochastic system can have different qualities. A system may be stable in first moment but not mean square stable. Due to proposition 2.1, quality of stability for a system that is stable in probability can be studied by finding h. For example, a faster growth of h can imply a better convergence. Therefore, the h-mean stability is a strong and exact stability notion.

3. Stability Theorem

In this section the main results of the paper is presented. First a Lyapunov theorem is proposed for the delayed system (1). Then, the theorem is rewritten for the special case of a delay free system.

3.1 Delayed systems

The Lyapunov stability theorem for (1) is as below.

Theorem 3.1: The system (1), is w_1-mean stable if there is a differentiable functional V: $C^n(\bar{\mathbb{R}}^+)\times\mathbb{R}\rightarrow\bar{\mathbb{R}}^+$ satisfying (5) for some $w_1,w_2\in K_d$ and there exist $r\in\bar{\mathbb{R}}^+$ such that $E\{{}^d/_{dt}V(x_t,t)\}$ is well-defined and non-positive for every $\|x_{t_0}\|_\infty< r$, $t\geq t_0$. Also, (1) is asymptotically h-mean stable for $h\in K_d$ if there exist $u\in K_d$ such that $\|x_{t_0}\|_\infty< r$ implies $E\{{}^d/_{dt}V(x_t,t)\}\leq -E\{u(\|x(t)\|)\}$ for every $t\geq t_0$ and h is covered by both u and w_1.

$$w_1(\|x(t)\|)\leq V(x_t,t)\leq w_2(\|x\|_\infty) \tag{5}$$

Proof: According to Equation (1), every sample path of x is differentiable with respect to t. Hence, due to differentiability of V, the time derivative of V exists and it can be easily shown that the time derivation operator d/dt commutes with the expectation operator E as below.

$$E\left\{\frac{d}{dt}V(x_t,t)\right\}=E\left\{\lim_{s\to t}\frac{V(x_s,s)-V(x_t,t)}{s-t}\right\}=$$
$$\lim_{s\to t}\frac{E\{V(x_s,s)\}-E\{V(x_t,t)\}}{s-t}=\frac{d}{dt}E\{V(x_t,t)\}\quad\Rightarrow$$
$$\frac{d}{dt}E\{V(x_t,t)\}=E\left\{\frac{d}{dt}V(x_t,t)\right\} \tag{6}$$

Also, according to continuity of w_2, for every $\varepsilon>0$ we can select $0<\delta<r$ such that (7) is satisfied.

$$w_2(\delta)<\varepsilon \tag{7}$$

Theorem 3.1 has two parts, proved in the following.

Part 1: According to the selected $\delta<r$ and conditions of the theorem, if we select $\|x_{t_0}\|_\infty<\delta$ then $E\{{}^d/_{dt}V\}\leq 0$ and consequently ${}^d/_{dt}E\{V\}\leq 0$ due to (6). This implies non-increasing behavior of $E\{V\}$ with time. Using this fact, (5) and (7) one can write (8) which proves the first part according to definition 2.2.

$$E\{w_1(\|x(t)\|)\}\leq E\{V(x_t,t)\}\leq V(x_{t0},t_0)\leq w_2(\delta)<\varepsilon \tag{8}$$

Part 2: For ε and δ in (7) and $\|x_{t_0}\|_\infty<\delta$ we have (9) which implies decreasing behavior of $E\{V\}$ and (8) as in previous part of proof.

$$E\{{}^d/_{dt}V(x_t,t)\}<-E\{u(\|x(t)\|)\}<0 \tag{9}$$

Since $0\leq E\{V\}<\varepsilon$, its decreasing behavior implies that $m=\lim_{t\to\infty}E\{V\}$ is a constant between 0 and ε. Now we define $\hat{V}(x_t,t)=m+\int_t^\infty E\{u(\|x(s)\|)\}ds$. According to (9), it follows that $m\leq\hat{V}(x_t,t)\leq E\{V(x_t,t)\}$ and consequently $\lim_{t\to\infty}E\{\hat{V}\}=$ m (using the squeeze lemma). By definition of $\hat{V}$ we have ${}^d/_{dt}\hat{V}=-E\{u(\|x(t)\|)\}$ and we can apply the Barbalat's lemma to conclude (10) as below. Because ${}^d/_{dt}\hat{V}$ is continuous with respect to time according to continuity of u and differentiability of x with respect to t in (1).

$$\lim_{t\to\infty}E\{u(\|x(t)\|)\}=0 \tag{10}$$

Since u covers h, there exist a constant $c_1\in\bar{\mathbb{R}}^+$ such that $0<c_1h(\|x\|)<u(\|x\|)$. Taking expectation from this inequality and tending t to infinity we obtain $\lim_{t\to\infty}E_t\{h(\|x(t)\|)\}=0$ according to (10). This fact together with Lemma 3.1 in the following proves the second part. □

Lemma 3.1: if (1) is w-mean stable for some $w\in K_d$ then it is h-mean stable for every $h\in K_d$ that is covered by w.

Proof: The w-mean stability of (1) can be written as (11) according to definition 2.2.

$$\forall\bar{\varepsilon}>0,\exists\delta>0\mid\|x_{t_0}\|<\delta\Rightarrow E_t\{w(\|x(t)\|)\}<\bar{\varepsilon} \tag{11}$$

There exist $c_1\in\bar{\mathbb{R}}^+$ such that $w(\|x\|)>c_1h(\|x\|)$. Combining this inequality with consequent part of (11) and setting $\bar{\varepsilon}$ to $c_1\varepsilon$ for an arbitrary $\varepsilon>0$, one can write the following and obtain (12) which is equivalent to h-mean stability of (1) according to definition 2.2.

$$\forall\varepsilon>0,\exists\delta>0\mid\|x_{t_0}\|<\delta\Rightarrow$$
$$c_1E_t\{h(\|x(t)\|)\}<E_t\{w(\|x(t)\|)\}<\bar{\varepsilon}=c_1\varepsilon$$
$$\forall\varepsilon>0,\exists\delta>0\mid\|x_{t_0}\|<\delta\Rightarrow E_t\{h(\|x(t)\|)\}<\varepsilon \tag{12}$$

The h-mean stability notion has a natural relationship with theorem 3.1, which results in shortening the proof of theorem. This fact and remark 2.4, are main reasons of using h-mean stability notion in this work.

Remark 3.1: For every set $U=\{u_i\in K_d:1\leq i\leq n\}$ of K_d functions, there always exist functions that are covered by all elements of U. An example is $h_1(\alpha)=\inf_i\{b_iu_i(\alpha)\}$ where b_i $(1\leq i\leq n)$ are arbitrary positive real numbers (since $u_i\geq b_i^{-1}h_1$). This ensures that in second part of theorem 3.1 there always exists a function h that is covered by u and w_1.

Remark 3.2: The stochastic control theory ([5,7]) is mainly concerned with systems modeled by Itô SDEs (remark 2.1). As a result, second derivatives of V appear in calculation of $E_t\{{}^d/_{dt}V\}$. In this work, no assumption is made about θ and $E_t\{{}^d/_{dt}V\}$ is not calculated. As a result, theorem 3.1 is applicable to problems that are different from the problems commonly modeled by the Itô SDE. An application is the case of next section. Other applications may include randomly switched systems [8].

3.2 Delay free systems

State vector $x(t)$ is a part of history x_t. Hence, the delay free system (13) is a special case of delayed system (1).

$$\dot{x}(t)=f\big(x(t),t,\theta(t)\big) \tag{13}$$

Stability definitions 2.2 and 2.3 are written for (13) by replacing $\|x_{t_0}\|$ in antecedents of (3) and (4) with $\|x(t_0)\|$.

Accordingly, theorem 3.1 in previous part is simplified to theorem 3.2 for the delay free system (13).

Theorem 3.2: System (13), is w_1-mean stable if there exist a function V: $\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^+$+ and $w_1,w_2\in K_d$ such that (14) is satisfied and there exist $r\in\mathbb{R}^+$ such that $\|x(t_0)\| < r$ implies $E\{{}^d/_{dt}V(x(t),t)\} \le 0$ for every $t \ge t_0$. Additionally, for $h\in K_d$ (1) is asymptotically h-mean stable if there exist $u\in K_d$ such that $\|x(t_0)\| < r$ implies $E\{{}^d/_{dt}V(x(t),t)\} \le -E\{u(\|x(t)\|)\}$ for every $t \ge t_0$ and h is covered by both u and w_1.

$$w_1(\|x\|) \le V(x,t) \le w_2(\|x\|) \tag{14}$$

4. Application to NCS Problem

In this section, a nonlinear NCS problem will be studied which has been originally proposed in [17]. The configuration of this NCS is depicted in Fig.1. In [17] it is assumed that there is no communication delay and the goal is to obtain a bound on maximum allowed time interval between data transmissions that can guarantee the stability of NCS. This bound is known as MATI (maximum allowable transfer interval).

Using Theorem 3.1, the problem can be extended from two different aspects. First, instead of finding a bound, we will be able to check stability when some probabilistic data about the transfer intervals is available as a PDF. Second, it will be possible to handle an NCS with communication delays. However, due to complexities of the extension to delayed case and the limited space, we will only focus on the extension to the probabilistic case in this work.

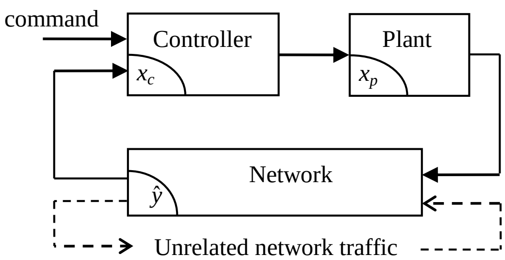

Fig 1. The networked control system (NCS).

The plant and controller can be modeled as (15) and (16) respectively in which x_p is the state of plant, x_c is the controller state, u_p is the plant input, y is the plant output and $\hat{y} = y(\hat{t})$ is the latest sample of y available at controller which is obtained at sampling instant $\hat{t}$ ([17]).

$$\dot{x}_p = f_p(x_p, u_p, t),\ \ y = g_p(x_p, t) \tag{15}$$

$$\dot{x}_c = f_c(x_c, \hat{y}, t),\ \ \ u_p = g_c(x_c, \hat{y}, t) \tag{16}$$

The above equations can be combined during time interval that $\hat{y}$ is constant (no updated data is received). The result is (17) in which $x^T=[x_p^T\ \ x_c^T]$, $\hat{x} = x(\hat{t})$ and the network induced error is defined as $e = x - \hat{x}$.

$$\dot{e} = f(e, \hat{x}, t) \tag{17}$$

For simplicity, we will consider the situation where all feedback data is transmitted at once. This is the case for example when the plant is single-input single-output. However, the results can be extended to the case of multiple transmitters. It is assumed that a continuously differentiable and positive definite Lyapunov function $V(x,t)$ exists that satisfies (18) to (20) globally with positive real numbers c_1, c_2, c_3, c_4.

$$c_1\|x\|^2 \le V(x,t) \le c_2\|x\|^2 \tag{18}$$

$$\frac{\partial}{\partial t}V + \frac{\partial}{\partial x}V\, f(0,x,t) \le -c_3\|x\|^2 \tag{19}$$

$$\left\|\frac{\partial}{\partial x}V\right\| \le c_4\|x\| \tag{20}$$

The functions f and g are also assumed to be globally Lipschitz such that one can write (21).

$$\|f(e,\hat{x},t)\| \le k_1\|e\| + k_2\|\hat{x}\| \tag{21}$$

$$\|f(e, x-e, t) - f(0,x,t)\| \le k_p\|e\| \tag{22}$$

Time derivative of V can be calculated as below using (19), (20) and (21-1).

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\,\dot{x} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\, f(e,\hat{x},t) =$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\, f(0,x,t) + \frac{\partial V}{\partial x}\,[f(e,x-e,t) - f(0,x,t)]$$

$$\dot{V} \le -c_3\|x\|^2 + c_4\|x\|k_p\|e\|$$

Using triangle inequality we have

$$\dot{V} \le -c_3[\|\hat{x}\| - \|e\|]^2 + c_4(\|\hat{x}\| + \|e\|)k_p\|e\|$$

$$\dot{V} \le -c_3\|\hat{x}\|^2 + (c_4k_p + 2c_3)\|\hat{x}\|\|e\| + (c_4k_p - c_3)\|e\|^2 \tag{23}$$

Using (17) and (21), we can write

$$\frac{d}{dt}\|e\| \le \|\dot{e}\| = \|f(e,\hat{x},t)\| \le k_1\|e\| + k_2\|\hat{x}\|$$

Using the comparison lemma ([23]) we obtain an upper bound on $\|e\|$ as below

$$\|e\| \le \frac{[e^{k_1\Delta t} - 1]k_2}{k_1}\|\hat{x}\| \tag{24}$$

$$\Delta t = t - \hat{t}$$

In the same way, we can obtain a differential inequality and solve in the reverse direction of time from t to $\hat{t}$ to obtain the following bounding

$$\|\hat{x}\| \le \frac{[e^{(k_1+k_2)|\Delta t|} - 1]k_1}{k_1 + k_2}\|x\| \tag{25}$$

Relations (23) and (24) give an upper bound on $\dot{V}$

$$\dot{V} \le \phi_a(\Delta t)\|\hat{x}\|^2 \tag{26}$$

$$\phi_a(\Delta t) = -c_3 + (c_4k_p + 2c_3)\frac{[e^{k_1\Delta t} - 1]k_2}{k_1} + |c_4k_p - c_3|\left[\frac{[e^{k_1\Delta t} - 1]k_2}{k_1}\right]^2 \tag{27}$$

For every i, the ith transfer interval is denoted by T_i. In deterministic case, $V \leq 0$ guarantees stability due to the common Lyapunov theorem. According to (26) $V \leq 0$ is resulted from $\phi_a(\Delta t) \leq 0$.

The function, $\phi_a(s)$ is increasing with $\phi_a(0) < 0$. Therefore, $\phi_a(s)=0$ has a unique positive solution τ_a which is a lower bound for MATI because $T_i \leq \tau_a$ implies $\phi_a(\Delta t) \leq 0$. This is similar to the results in [17]. But how would be the NCS stability if T_i can exceed τ_a. To answer such a question we can use theorem 3.2 in previous section. Taking expectation from (25) we have (28).

$$E\{\dot{V}\} \leq E\{\phi_a(\Delta t)\}\|\hat{x}\|^2 \tag{28}$$

Also we can obtain (29) from (26) by a few manipulations and taking expectation.

$$E\{\phi_b(\Delta t)\}\|\hat{x}\|^2 \leq E\{\|x\|^2\} \tag{29}$$

$$\phi_b(\Delta t) = \left[\frac{\left[e^{(k_1+k_2)|\Delta t|} - 1\right]k_1}{k_1 + k_2}\right]^{-2}$$

Now we can apply Theorem 3.2 as explained in the following. Due to (18), Lyapunov function V satisfies the condition (14) of Theorem 3.2 with $w_1(\alpha) = c_1\alpha^2$ and $w_2(\alpha) = c_2\alpha^2$. Therefore, the NCS is asymptotically mean square stable if $E\{\dot{V}\} \leq -c_5 E\{\|x\|^2\}$ for some $c_5 > 0$. But, if $E\{\phi_a(\Delta t)\}<0$ then we can combine (28) and (29) to obtain a positive constant $c_5 = -E\{\phi_a(\Delta t)\} / E\{\phi_b(\Delta t)\}$. However, for every i, $E\{\phi_a(\Delta t)\}<0$ is satisfied if $E_{T_i}\{\phi_a(T_i)\} < 0$ because ϕ_a is increasing and $\Delta t \leq T_i$. The result can be summarized as following corollary.

Corollary 4.1: NCS (15), (16) is asymptotically mean square stable (asymptotically stable in second moment) if the transfer intervals T_i from y to $\hat{y}$ have a common PDF denoted by p_T and there exists a Lyapunov function V for the closed loop system with $\hat{y} = y$ that satisfies (18), (19), (20), and the following condition is satisfied with ϕ_a defined in (26).

$$E\{\phi_a(T_i)\} = \int_0^\infty p_T(s)\,\phi_a(s)ds < 0 \tag{30}$$

Remark 4.1: If the random variations of transfer intervals T_i are due to data packet losses (packet transmission errors) with probability p_e and the sampling period is equal to h, then we have (31) in which δ is the Dirac's delta function.

$$p_T(s) = \sum_{i=1}^{\infty}(1-p_e)p_e^{i-1}\delta(s-ih) \tag{31}$$

Replacing (27) and (31) in the left hand side of (30), eliminating the Delta function and integration we obtain

$$E\{\phi_a(T_i)\} = \left(c_4k_p + 2c_3\right)\frac{k_2}{k_1}\sum_{i=1}^{\infty}(1-p_e)p_e^{i-1}\left[e^{k_1 ih} - 1\right]$$
$$+\left|c_4k_p - c_3\right|\left[\frac{k_2}{k_1}\right]^2\sum_{i=1}^{\infty}(1-p_e)p_e^{i-1}\left[e^{k_1 ih} - 1\right]^2 - c_3$$

The right hand side of the above equation contains two geometric series with common ratios $p_e \exp(k_1h)$ and $p_e \exp(2k_1h)$ that must be smaller than one to ensure the convergence. Since the later one is always greater, it suffices to have $p_e < \exp(-2k_1h)$. Simplifying the result, the condition (30) can be represented as below

$$p_e < e^{-2k_1h} \tag{32-1}$$

$$E\{\phi_a(T_i)\} = \beta_0 + \beta_1\frac{e^{k_1h}(1-p_e)}{1-e^{k_1h}p_e} + \beta_2\frac{e^{2k_1h}(1-p_e)}{1-e^{2k_1h}p_e} < 0 \tag{32-2}$$

$$\beta_0 = -c_3 - \left(c_4k_p + 2c_3\right)\frac{k_2}{k_1} + \left|c_4k_p - c_3\right|\left[\frac{k_2}{k_1}\right]^2$$

$$\beta_1 = \left(c_4k_p + 2c_3\right)\frac{k_2}{k_1} - 2\left|c_4k_p - c_3\right|\left[\frac{k_2}{k_1}\right]^2$$

$$\beta_2 = \left|c_4k_p - c_3\right|\left[\frac{k_2}{k_1}\right]^2$$

which can be calculated to rewrite (30) as below provided that $p_e < \exp(-2k_1h)$.

Example 4.1: The following NCS is considered.

$$\dot{x}_1 = x_2 - x_1$$
$$\dot{x}_2 = (x_2 + x_1)\sin x_1 + u$$
$$y = x_2$$
$$u = -3\hat{y}$$

A Lyapunov function for the closed loop with $\hat{y} = y$ can be $V=½(x_1^2 + x_2^2)$ with $c_1 = c_2 = ½$, $c_3 = 0.38$, $c_4 = 1$ which guarantees deterministic stability globally. Equations (17) can be written as below.

$$\dot{e}_1 = \hat{x}_2 - \hat{x}_1 + e_2 - e_1$$
$$\dot{e}_2 = (\hat{x}_2 + \hat{x}_1 + e_2 + e_1)\sin(\hat{x}_1 + e_1) - 3\hat{x}_2$$

Using the above equations, we can obtain k_1=k_2=3.34 and k_p=3. This completely determines the function ϕ_a in (26). The obtained ϕ_a gives τ_a = 0.0215 which guaranties the stability for $T_i \leq 0.0215$.

Many random communication effects can be studied using Corollary 4.1. It is assumed that we have transmissions with possibility of error as explained in Remark 4.1. In Figure 2 the maximum value of p_e that satisfies (32) is plotted as a function of h for the obtained ϕ_a. This plot gives a lower bound for the stability margin of the packet loss probability p_e.

5. A Practical NCS Application

In this section we study a practical NCS that consists of a dual axis magnetic levitation system controlled over a communication link to a computer. In the following, in the first part we describe the control system. In the second part we describe the communication system and its limitations. In the third and last part we use the results of this paper to select the communication data rate such that the stochastic stability of the control system is preserved.

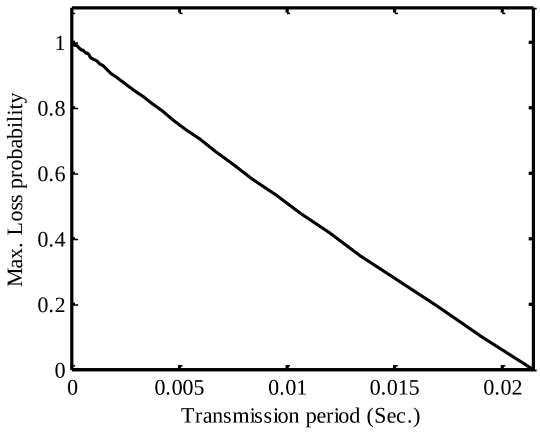

Fig. 2. Transmission period h versus loss probability p_L

5.1 Control system

The magnetic levitation system is composed of a steel ball with mass m_b=40 g affected by two magnetic forces F_x, F_y generated by two identical solenoids with voltages v_x, v_y and currents i_x, i_y respectively as shown in Figure 3 (a). The forces acting on the ball are Fx, F_y and the weight of ball $m_b g$ (g = 9.8 m/sec.^{2} is the acceleration of gravity) as depicted in Figure 3 (b).

Assuming that the axes of coils (x and y axes) are perpendicular, the magnetic forces F_x, F_y can be calculated from the following equations in which K_f=0.0047 Kg m^3/C^2 is magnetic force constant.

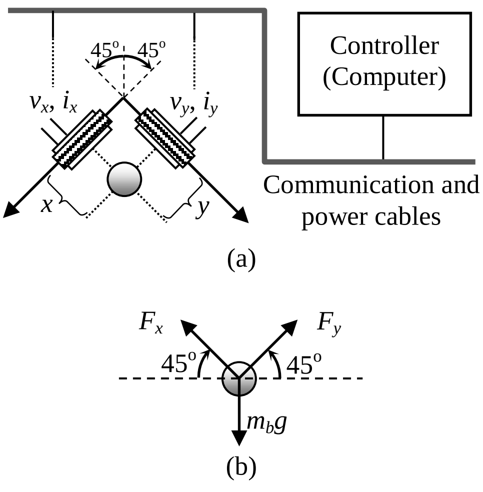

Fig. 3. (a) The magnetic levitation control system. (b) The forces acting on the ball.

$$F_x = K_f \left({}^{i_x}/_{x} \right)^2, \quad F_y = K_f \left({}^{i_y}/_{y} \right)^2 \tag{33}$$

The resistance and inductance of the coils are R=0.62 Ω and L=0.32 H such that we can write

$$v_x = R\, i_x + L\, \dot{i}_x, \quad v_y = R\, i_y + L\, \dot{i}_y \tag{34}$$

The equations of motion of the ball are also as below.

$$m_b \ddot{x} = m_b g \sqrt{2}/2 - F_x, \qquad m_b \ddot{y} = m_b g \sqrt{2}/2 - F_y \tag{35}$$

The control algorithm is implemented in a computer (Figure 1) that receives the measurement feedbacks i_x, i_y, x, y, $\dot{x}$, $\dot{y}$ from the node which is connected to the coils and sends back the control commands v_x and v_y to the coils.

The control commands to the input voltages v_x and v_y are calculated using the feedback linearization method as follows. First we differentiate the equations in (35) and summarize the results as in the following two equations (detailed representation of functions f_a and g_a are omitted for brievity).

$$\dddot{x} = f_a(x, \dot{x}, i_x) + g_a(x, \dot{x}, i_x) v_x, \qquad \dddot{y} = f_a(y, \dot{y}, i_y) + g_a(y, \dot{y}, i_y) v_y \tag{36}$$

Based on the above equations we design the control laws in (37) to achieve the closed loop transfer functions in (38) where x_d and y_d are the desired values for x and y.

$$v_x = \frac{-\alpha_1 \ddot{x} - \alpha_2 \dot{x} - \alpha_3 (x - x_d) - f_a(x, \dot{x}, i_x)}{g_a(x, \dot{x}, i_x)}, \qquad v_y = \frac{-\alpha_1 \ddot{y} - \alpha_2 \dot{y} - \alpha_3 (y - y_d) - f_a(y, \dot{y}, i_y)}{g_a(y, \dot{y}, i_y)} \tag{37}$$

$$\frac{X(s)}{X_d(s)} = \frac{Y(s)}{Y_d(s)} = \frac{1}{s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3} \tag{38}$$

The values of $\ddot{x}$ and $\ddot{y}$ in (37) are obtained from equations (33) through (35) in terms of the measured variables.

The controller parameters are selected as α_1 = 4.47, α_2 = 8.64, α_3 = 6.1 and the control objective is to maintain the ball at position x_d=y_d = 0.65 m.

5.2 Communication

The solenoids require electric power which is supplied through power cables. To reduce wiring we would like to transmit the control data through the power cables. The transmission bit-rate is denoted by f_{tx}. The communication through the power cables suffers from the noise in an industrial environment (we assume that there is no bandwidth limitation). We denote the noise at the bit detection stage by $w(t)$ and assume that it is Gaussian with $E\{w(t)\}$ = 0, $E\{w^2(t)\} = \sigma_w^2$ = 1 and power spectral density $S_w(f)$ in (39) where f_b=10^{8}Hz is the noise bandwidth.

$$S_w(f) = \begin{cases} \sigma_w^2 / 2 f_b & |f| \le f_b \\ 0 & |f| > f_b \end{cases} \tag{39}$$

The signal level at receiver is assumed to be v_{bit}=1 v. In general, the reliability of data transmission increases if we reduce f_{tx}. For example with a smaller f_{tx}, we can decrease the bandwidth of the low-pass filter at the baseband processing stage in the receiver to reduce the effect of noise on the bit detection. We use a simple low-pass filter with transfer function $H_f(s)$ = 1/[1+s/(4πf_{tx})]. Then the power spectral density of the filtered noise $w_f(t)$ is

$$S_{w_f}(f) = \frac{1}{1 + (f/f_{tx})^2} S_w(f) \tag{40}$$

The variance (power) of $w_f(t)$ can be calculated as

$$\sigma_{w_f}^2 = f_{tx}\, tan^{-1}(f_b/f_{tx}) \quad (41)$$

Assuming that the sampling for bit detection is performed at the end of the bit hold time $1/f_{tx}$ and neglecting the effect of filter on the signal amplitude at the sampling instant, the probability of a bit transmission error p_{bit} is

$$p_{bit} = 1 - \frac{1}{2} erf\left(\frac{v_{bit}}{\sqrt{2}\,\sigma_{w_f}}\right) \quad (42)$$

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-s^2} ds$$

Consequently probability of a frame transmission error p_e is obtained as below where n_f is the length of frame in bits

$$p_e = 1 - (1 - p_{bit})^{n_f} \quad (43)$$

5.3 Analysis

In this part we study the interaction of the control system and communication described in the previous parts of this section to select bit-rate of communication f_{tx} such that the control loop remains stable.

Each control cycle begins with a sampling at sensors on the solenoids side, transmission of the measurement data through the coils-controller link to the control computer, execution of the control algorithm which is assumed to take τ_c = 50 μs and sending back the voltage commands through the controller-coils link to the solenoids. The sensor measurements include 6 values and control commands include 2 values as described previously. Assuming that each value is encoded in 10 bits and that the framing adds 10 extra bits as header, the transmission time of the measured values and command values are $\tau_s=70/f_{tx}$ and $\tau_a=30/f_{tx}$ respectively. We assume that the control loop is allowed to use 33 percents of the communication capacity (time division). Hence, the length of control cycle becomes

$$h = \tau_c + 3[\tau_s + \tau_a] = 300/f_{tx} + 50\times10^{-6} \quad (44)$$

Since the controller in (37) is static (it does not have states), we can concatenate the communication delays and assume that there is a single delay of length h during a control cycle.

Now, for a given value of transmission bit-rate f_{tx} we can obtain p_e and h from (43) and (44) and use condition (32) in remark 4.1 to calculate $E\{\phi_a(T_i)\}$ in Corollary 4.1 and check the stability. This is performed for a range of values for f_{tx} in Figure 4.

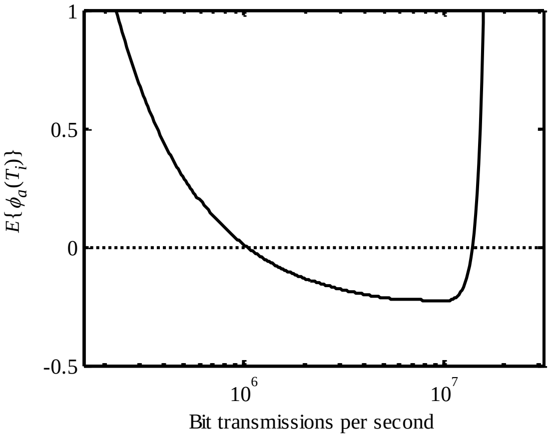

Fig. 4. (a) The magnetic levitation control system.

According to Figure 4, the stability of control loop is preserved between f_{tx}=1 MBPS (Mega bit per sec.) and f_{tx}=14 MBPS. A good selection is f_{tx}=5 MBPS which is sufficiently away from the stability margins and gives a sampling frequency of 9.1 KHz.

6. Conclusions

In this paper, we observed that the focus of the theory of stochastic systems has been centered on a special kind of approach to modeling the stochastic phenomena. Even if this approach has been sufficient for the past applications, with the growing complexity of the new systems, it is expectable that we will need to expand the capabilities of the existing analysis frameworks.

Based on this observation, a stochastic Lyapunov theorem was presented. This theorem benefits from a higher level of generality. This was shown by applying the theorem to a practical NCS problem. The result is a new stability analysis criterion for a stochastic nonlinear NCS that cannot be obtained using the traditional versions of the stochastic Lyapunov theorem. However, we had to carry out some additional calculations in section four in order to be able to apply the theorem from section three to the NCS problem. This seems to be the cost that we have to pay for the generality that we have obtained. There are more potential applications to the various NCS problems that will be investigated in future works.

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